Introductory Fourier Analysis

Joshua Nauman

September 2025

1 Motivation

For a function $f : \mathbb{R} \to \mathbb{R}$, we can write it as a Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where c_n depends on the derivatives of f at a. However, there exist functions $f(x)$ such that, for all $n \in \mathbb{N}$, $\partial_x^n f$ exist and are continuous, yet its Taylor Series converges to f nowhere. Take, for example, the following function

$$g(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Thus, we want another method of writing a series for a function, which is where the Fourier Series comes in. As it turns out, the Fourier Series, as well as its natural extension, the Fourier Transform, have many applications, from biology to music to partial differential equations.

2 Important Facts

The following integral identities will be important:

$$\int_{-a}^{a} \sin(mx)\cos(nx)\mathrm{d}x = 0 \qquad \forall m, n \tag{1}$$

$$\int_{-a}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = 0 \qquad m, n \in \mathbb{Z}; m \neq n; a \in \mathbb{R} \tag{2}$$

$$\int_{-a}^{a} \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)\mathrm{d}x = 0 \qquad m, n \in \mathbb{Z}; m \neq n; a \in \mathbb{R} \tag{3}$$

$$\int_{-a}^{a} \sin^2\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \int_{-a}^{a} \cos^2\left(\frac{n\pi}{a}x\right)\mathrm{d}x = a \qquad n \in \mathbb{Z}; n \neq 0; a \in \mathbb{R} \tag{4}$$

2.1 Derivation of Identities

2.1.1 Identity 1

We wish to show that for any a, n, m

$$\int_{-a}^{a} \sin(nx)\cos(mx)\mathrm{d}x = 0.$$

Proof. Consider the case where $n = m$. Then we perform a substitution $u = \sin(nx)$ and $\mathrm{d}u = n\cos(nx)$ then we have

$$\int_{-a}^{a} \sin(nx)\cos(nx)\mathrm{d}x = \frac{1}{2n}\sin^2(nx)\Big|_{-a}^{a} = 0.$$

If $n \neq m$, then, since $\cos(y) = \cos(-y)$,

$$\begin{aligned}\int_{-a}^{a} \sin(nx)\cos(mx)\mathrm{d}x &= \int_{-a}^{a} \frac{1}{2}\left(\sin((n+m)x) + \sin((n-m)x)\right)\mathrm{d}x \\ &= -\frac{1}{2}\left(\frac{\cos((n+m)x)}{n+m} + \frac{\cos((n-m)x)}{n-m}\right)\Big|_{-a}^{a} \\ &= -\frac{1}{2}\left(\frac{\cos((n+m)a)}{n+m} - \frac{\cos(-(n+m)a)}{n+m} + \frac{\cos((n-m)a)}{n-m} - \frac{\cos(-(n-m)a)}{n-m}\right) \\ &= 0,\end{aligned}$$

where we used the sum and difference formulas for sine to derive the first line. □

2.1.2 Identity 2

We wish to show that for any integers m, n such that $m \neq n$ and for some $a \in \mathbb{R}$,

$$\int_{-a}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = 0.$$

Proof. Since $\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$, we can instead evaluate the equivalent integral,

$$\int_{-a}^{a} \frac{1}{2}\left(\cos\left(\frac{\pi}{a}x(n-m)\right) - \cos\left(\frac{\pi}{a}x(n+m)\right)\right) = \frac{a}{2\pi}\left(\frac{\sin\left(\frac{\pi}{a}x(n-m)\right)}{n-m} - \frac{\sin\left(\frac{\pi}{a}x(n+m)\right)}{n+m}\right)\Bigg|_{-a}^{a}.$$

Now, whether we plug in $-a$ or a, we get 0 because we have the sine of some integer multiple of π since $(n-m), (n+m) \in \mathbb{Z}$. □

2.1.3 Identity 3

We wish to show that for some integers m, n such that $m \neq n$ and for some $a \in \mathbb{R}$,

$$\int_{-a}^{a} \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)\mathrm{d}x = 0.$$

The proof of this will be very similar to the last.

Proof. Since $\cos A \cos B = \frac{1}{2}\big(\cos(A+B) + \cos(A-B)\big)$, we can instead evaluate the equivalent integral,

$$\int_{-a}^{a} \frac{1}{2}\left(\cos\left(\frac{\pi}{a}x(n+m)\right) + \cos\left(\frac{\pi}{a}x(n-m)\right)\right) = \frac{a}{2\pi}\left(\frac{\sin\left(\frac{\pi}{a}x(n+m)\right)}{n+m} + \frac{\sin\left(\frac{\pi}{a}x(n-m)\right)}{n-m}\right)\Bigg|_{-a}^{a}.$$

Again, whether we plug in $-a$ or a, since $(n+m), (n-m) \in \mathbb{Z}$, the sine will be 0. □

2.1.4 Identity 4

We wish to show that for some nonzero integer n and some real number a,

$$\int_{-a}^{a} \sin^2\left(\frac{n\pi}{a}x\right) \mathrm{d}x = \int_{-a}^{a} \cos^2\left(\frac{n\pi}{a}x\right) \mathrm{d}x = a.$$

Proof. Since $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ and by symmetry,

$$\int_{-a}^{a} \sin^2\left(\frac{n\pi}{a}x\right) \mathrm{d}x = \int_{0}^{a} 1 - \cos\left(\frac{2n\pi}{a}x\right) \mathrm{d}x = a - \sin(2n\pi) + \sin(0) = a.$$

Similarly, since $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ and again by symmetry of the region,

$$\int_{-a}^{a} \cos^2\left(\frac{n\pi}{a}x\right) \mathrm{d}x = \int_{0}^{a} 1 + \cos\left(\frac{2n\pi}{a}x\right) \mathrm{d}x = a + \sin(2n\pi) - \sin(0) = a.$$

In each case, $\sin(2n\pi) = 0$ since $n \in \mathbb{Z}$. Since we have $2n\pi$ inside the sine and sine is 0 for any integer multiple of π, this formula holds for some special $n \in \mathbb{Q} \setminus \mathbb{Z}$, such as $n = \frac{1}{2}$. $\square$

3 Inner Products

Inner Products are a generalization of the dot product. The dot product is defined as a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is, it defines vector multiplication by taking 2 vectors in $\mathbb{R}^n$ and mapping them to a real number. The dot product satisfies a few important properties. For $u, v, w, x \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$,

- $(u + v) \cdot (w + x) = u \cdot w + u \cdot x + v \cdot w + v \cdot x$.
- $(au) \cdot (bv) = ab(u \cdot v)$.
- $u \cdot v = v \cdot u$.
- $v \cdot v \geq 0$ and equal if and only if $v = 0$.

An inner product has a similar motivation. An inner product is any function $f : V \times V \to \mathbb{F}$, where V is a vector space and $\mathbb{F}$ is a field, that satisfies similar properties, defined shortly. First, we denote the inner product of vectors u and v as $\langle u, v\rangle$. In the case of the dot product in $\mathbb{R}^n$, $u \cdot v = \langle u, v\rangle = \displaystyle\sum_{i=1}^{n} u_i v_i$ where u_i and v_i represent the ith entry in the vector u and v, respectively. Let $u, v, w, x \in V$ for a vector space V over $\mathbb{C}$ and let $a, b \in \mathbb{C}$. An inner product is an operation satisfying the following:

- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.
- $\langle au, bv\rangle = a\bar{b}\langle u, v\rangle$.
- $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
- $\langle v, v\rangle \geq 0$ and equal if and only if $v = 0$.

Here, $\overline{z}$ refers to complex conjugation (ie. $\overline{a+bi} = a - bi$). For a vector space over the real numbers, these conditions simplify. Let $u, v, w, x \in W$ for a vector space W over $\mathbb{R}$ and let $a, b \in \mathbb{R}$. An inner product is an operation satisfying the following:

- $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.
- $\langle au, bv\rangle = ab\langle u, v\rangle$.
- $\langle u, v\rangle = \langle v, u\rangle$.
- $\langle v, v\rangle \geq 0$ and equal if and only if $v = 0$.

The first axiom tells us that $\langle u+v, w+x\rangle = \langle u, w\rangle + \langle u, x\rangle + \langle v, w\rangle + \langle v, x\rangle$. Over $\mathbb{C}$, it also turns out that this rule holds as well.

Proof. Note that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. For vectors $u, v, w, x \in V$ for a vector space V over $\mathbb{C}$,

$$\begin{aligned}\langle u+v, w+x\rangle = \langle u, w+x\rangle + \langle v, w+x\rangle = \overline{\langle w+x, u\rangle} + \overline{\langle w+x, v\rangle} \\ = \overline{\langle w, u\rangle + \langle x, u\rangle} + \overline{\langle w, v\rangle + \langle x, v\rangle} = \overline{\langle w, u\rangle} + \overline{\langle x, u\rangle} + \overline{\langle w, v\rangle} + \overline{\langle x, v\rangle} \\ = \langle u, w\rangle + \langle u, x\rangle + \langle v, w\rangle + \langle v, x\rangle.\end{aligned}$$

□

There is much more to inner products, but this is more than what we will need to define the L^2 inner product, which defines multiplication of vectors in a vector space of real functions. For f, g real functions of x in a vector space of real functions, V, the L^2 inner product is given by

$$\langle f, g\rangle_{L^2} = \int_V f\ g\ \mathrm{d}x.$$

If the domain of the functions in this vector space is the closed interval $[a, b]$, the integral becomes

$$\langle f, g\rangle_{L^2} = \int_a^b f\ g\ \mathrm{d}x.$$

Now, consider the infinite dimensional function space $L^2([-L, L])$, the set of all Lebesgue integrable functions $f(x)$ defined over $[-L, L]$, that is, functions $f(x)$ defined over $[-L, L]$ such that

$$\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x < \infty,$$ [1]

with the a basis of infinite length $\left\{1, \cos\left(\frac{m\pi}{L}x\right), \sin\left(\frac{m\pi}{L}x\right)\ \middle|\ m \in \mathbb{N}\right\}$. Then, to write a function $f \in L^2([-L, L])$ as a linear combination of basis vectors, we must project f onto each of the basis elements. In $\mathbb{R}^2$ with standard basis e_1, e_2, a vector $v \in \mathbb{R}^2$ has a component in the e_1 direction given by the projection of v onto e_1,

$$v_{e_1} = \frac{v \cdot e_1}{e_1 \cdot e_1} e_1,$$

[1] This is actually a slight simplification. As above, we speak of functions $f : [-L, L] \to \mathbb{C}$ that are square Riemann integrable. The vector space $L^2([-L, L])$ of square Lebesgue integrable functions is actually larger. This does not affect further results. The reason for the absolute values is to properly handle complex-valued functions.

and component in the e_2 direction given similarly by

$$v_{e_2} = \frac{v \cdot e_2}{e_2 \cdot e_2} e_2.$$

In $L^2([-L, L])$, instead of the dot product, we are equipped with the inner product defined above as $\langle f, g\rangle_{L^2} = \displaystyle\int_{-L}^{L} f\ g\ \mathrm{d}x$. It follows then that to find the component of f in the direction of a given basis element ϕ by

$$f_\phi = \frac{\langle f, \phi\rangle_{L^2}}{\langle \phi, \phi\rangle_{L^2}} \phi.$$

This allows us to find the coefficient for any basis element so that we can write f as a linear combination of the basis elements. However, with our basis as it is currently defined, the elements are only orthogonal to one another in space (their inner products with one another will be 0), but they are not *normed* in the sense that their inner product with themselves will be 1. For example, we have, by (2) and (3),

$$\begin{aligned}
\langle \cos\left(\tfrac{m\pi}{L}x\right), \cos\left(\tfrac{m\pi}{L}x\right)\rangle_{L^2} &= L \\
\langle \sin\left(\tfrac{m\pi}{L}x\right), \sin\left(\tfrac{m\pi}{L}x\right)\rangle_{L^2} &= L \\
\langle 1, 1\rangle_{L^2} = \int_{-L}^{L} \mathrm{d}x &= 2L.
\end{aligned}$$

This will come in handy shortly. For ease of writing, call the first element of this basis ϕ_1, the mth cosine term ϕ_{mc}, and the mth sine term ϕ_{ms}. For example, $\phi_{4c} = \cos\left(\dfrac{4\pi}{L}x\right)$.

4 Fourier Series Definition

To extend a function $g : [-L, L] \to \mathbb{R}$ to a function that is $2L$-periodic (a period of length $2L$, for example $L = \pi$ and $f(x) = \sin(x)$), that is, construct a function $f : \mathbb{R} \to \mathbb{R}$ such $f(x) = g(x)$ for $x \in [-L, L]$, and extended to $\mathbb{R}$ by periodicity: $f(x + 2Lk) = f(x)$ for each $k \in \mathbb{Z}$, take

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\mathrm{d}x,$$

and

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x,$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

5 Derivation of a_n, b_n Formulae

Consider a function $f(x) : [-L, L] \to \mathbb{R}$ so that $f \in L^2([-L, L])$. We will extend this to a function $f : \mathbb{R} \to \mathbb{R}$ defined as a linear combination of the basis elements of the function space defined in the last section,

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right).$$

While this may not look exactly like a linear combination of the basis elements (it appears to be missing a constant times ϕ_1), plugging in $n = 0$, we see that we have an a_0 term times no trig function, thus a_0 is the coefficient of the basis element $\phi_1 = 1$. Essentially, this simplifies to the formula provided for arbitrary $f(x)$ in the last section. By finding proper coefficients a_0, a_n, b_n, this will agree with and lead to our initial definition. To do this, we must find formulas for a_0, a_n, and b_n (we need not consider the case of b_0 since, plugging in $n = 0$, we get a $b_0 \sin(0) = 0$ term, meaning b_0 has no effect). To do this, lets first consider the projection of f onto the basis element of $L^2([-L, L)]$, ϕ_{mc}, for some fixed m. This gives us

$$\frac{\langle f, \phi_{mc}\rangle_{L^2}}{\langle \phi_{mc}, \phi_{mc}\rangle_{L^2}} = \int_{-L}^{L} \left(\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) \frac{1}{L} \cos\left(\frac{m\pi}{L}x\right) \mathrm{d}x,$$

because the inner product in the denominator will equal L. Note that, by (1), for each n, the product of the sine terms and the $\cos\left(\frac{m\pi}{L}x\right)$ term will be 0. This reduces the integral to

$$\frac{1}{L}\int_{-L}^{L} \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) \mathrm{d}x.$$

Now, by (3), each cosine product will be 0 except for the case where $m = n$. Then, the product within the integral is completely reduced and evaluated using (4),

$$\frac{a_m}{L}\int_{-L}^{L} \cos^2\left(\frac{m\pi}{L}x\right) \mathrm{d}x = a_m.$$

This then implies that $a_m = \frac{1}{L}\langle f, \cos\left(\frac{m\pi}{L}x\right)\rangle_{L^2} = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{m\pi}{L}x\right) \mathrm{d}x$. So, we are left with the following formula:

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{m\pi}{L}x\right) \mathrm{d}x,$$

which is equivalent to that stated in our original definition, $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x$. By this exact same reasoning, one can show that

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

Now, we must consider our special case, a_0. To find a_0, note that, since a_0 is some constant times $1 = \phi_1$, a_0 is the coefficient of the basis element ϕ_1. To find this, we project f onto $\phi_1 = 1$. Since $\langle \phi_1, \phi_1\rangle = 2L$, we have

$$a_0 = \frac{\langle f, \phi_1\rangle_{L^2}}{\langle \phi_1, \phi_1\rangle_{L^2}} = \frac{1}{2L}\int_{-L}^{L} f(x) \mathrm{d}x.$$

We have now derived all 3 formulas presented in section 4.

6 Complex Fourier Series

Up until this point, we have been working over the real numbers. We will now work to extend this definition of Fourier Series to complex valued functions. Let's begin with our formula for a function as a Fourier Series over $\mathbb{R}$ that we began with in the last section:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right).$$

We begin by changing each trigonometric function to its exponential form using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \implies \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

For ease of writing, we will let $y = \frac{n\pi}{L}x$. Then,

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(y) + b_n \sin(y) = \sum_{n=0}^{\infty} a_n \frac{e^{iy} + e^{-iy}}{2} + b_n \frac{e^{iy} - e^{-iy}}{2i} = \frac{1}{2}\sum_{n=0}^{\infty} a_n(e^{iy} + e^{-iy}) - ib_n(e^{iy} - e^{-iy}).$$

Then, distributing the a_n's and b_n's and grouping the terms by exponential,

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} (a_n - ib_n)e^{iy} + (a_n + ib_n)e^{-iy}.$$

Now, by defining c_n as follows, we can change the bounds of the summation to be $(-\infty, \infty)$ and simplify this to only having one exponential in this summation,

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ a_n & n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \end{cases} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\mathrm{d}x,$$

which allows us to define the Complex Fourier Series for a function $f : [-L, L] \to \mathbb{C}$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{iy} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}.$$

7 Fourier Series Examples

7.1 Example 1

Find the Fourier Series for the function $f(x)$ where

$$f(x) = \begin{cases} x & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}, \qquad f(x + 2\pi) = f(x).$$

We know that our Fourier Series for f will be of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

since $L = \pi \implies \frac{\pi}{L} = 1$. We first find a_0,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x\,\mathrm{d}x + \int_{0}^{\pi} 0\,\mathrm{d}x\right) = -\frac{\pi}{4}.$$

To find the a_n's,

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,\mathrm{d}x &= \frac{1}{\pi}\left(\cancelto{0}{\frac{x}{n}\sin(nx)} + \frac{1}{n^2}\cos(nx)\right)\Bigg|_{-\pi}^{0} \\ &= \frac{1}{\pi n^2}\left(\cos(0) - \cos(n\pi)\right) = \frac{1}{\pi n^2}\left(1 + (-1)^{n+1}\right). \end{aligned}$$

To find the b_n's,

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,\mathrm{d}x &= \frac{1}{\pi}\left(\frac{-x}{n}\cos(nx) + \cancelto{0}{\frac{1}{n^2}\sin(nx)}\right)\Bigg|_{-\pi}^{0} \\ &= \frac{-1}{n}\cos(n\pi) = \frac{1}{n}(-1)^{n+1}. \end{aligned}$$

Thus,

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx).$$

We can verify this with a graph of the Fourier Series of f:

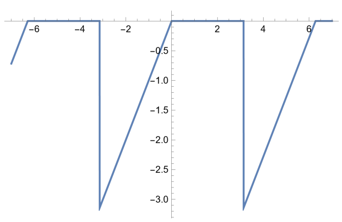

Figure 1: A plot of the Fourier Series for f for $x \in [-7, 7]$.

7.2 Example 2

Find the Fourier Series for $f(x)$ where

$$f(x) = \begin{cases} 0 & -\pi \le x < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le x < \pi \end{cases}, \qquad f(x + 2\pi) = f(x).$$

We know that the Fourier Series for f will look like

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx),$$

again because $L = \pi \implies \frac{\pi}{L} = 1$. Again, we start by finding a_0,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}(\pi) = \frac{1}{2}.$$

We now find the a_n's as usual,

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(nx)\,\mathrm{d}x = \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right).$$

When n is even, we will have the sine of some integer multiple of π, so $a_n = 0$ for even n. Thus, we can write a_n as

$$a_n = \begin{cases} \frac{2}{\pi n}(-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

In the case of the b_n's, they will all be 0 since $\displaystyle\int_{-\pi/2}^{\pi/2} \sin(nx) = 0$ for $n \in \mathbb{Z}$.

Since our sum will be 0 for even n, we can skip them by replacing n with $2n-1$ (or some other odd generating formula, adjusting the lower bound accordingly). Then,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{\pi(2n-1)} \cos((2n-1)x).$$

Again, we verify this with a graph,

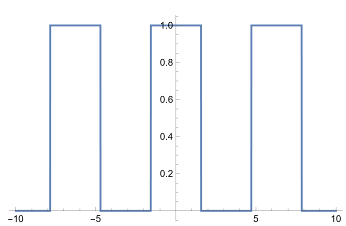

Figure 2: A plot of the Fourier Series for f for $x \in [-10, 10]$.

7.3 Example 3

Consider the aperiodic function $f(x) = x^2$ for $x \in [-2, 2]$ (assume f is 0 everywhere else). Extend $f : [-2, 2] \to \mathbb{R}$ to a function $g : \mathbb{R} \to \mathbb{R}$ by periodicity (that is, $g(x + 4k) = f(x)$ for $x \in [-2, 2]$ and $k \in \mathbb{Z}$. Here, $L = 2$, so we know that $g(x)$ will look like

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}x\right) + b_n \sin\left(\frac{n\pi}{2}x\right).$$

We begin as usual, finding a_0:

$$a_0 = \frac{1}{4} \int_{-2}^{2} x^2\,\mathrm{d}x = \frac{4}{3}.$$

We now find the a_n's as usual. Using repeated integration by parts, we see that

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2 \cos\left(\frac{n\pi}{2}x\right)\, \mathrm{d}x = \frac{16(-1)^n}{n^2\pi^2}.$$

The b_n's will be 0 since we will be integrating an odd function over a symmetric interval. We can verify this using integration by parts,

$$b_n = \frac{1}{2}\int_{-2}^{2} x^2 \sin\left(\frac{n\pi}{2}x\right)\, \mathrm{d}x = 0.$$

Then, we can see that

$$g(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi}{2}x\right).$$

Again, we verify this with a graph:

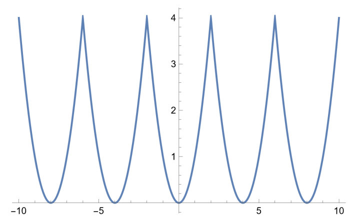

Figure 3: A graph of g, the Fourier Series for f, for $x \in [-10, 10]$.

8 Motivation for and Definition of the Fourier Transform

8.1 Motivation

While everything discussed above is extremely important and useful in its own right, it only applies for functions on $[-L, L]$. Naturally, we want to extend this method in such a way so that we can study a function defined on $\mathbb{R}$. This is where the Fourier Transform comes in. We will also come to find that the Fourier Transform has many properties that can be extremely helpful in certain situations, such as in solving differential equations.

Because we are considering a function defined over $\mathbb{R}$ instead of $[-L, L]$, the Fourier Transform is a natural extension of Fourier Series. We can think of the Fourier Transform as the coefficient c_n as $L \to \infty$ for a Fourier Series.

8.2 Definition

Stated plainly, for a square Lebesgue integrable function $f : \mathbb{R} \to \mathbb{C}$, the Fourier Transform of $f(x)$ is given by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}\mathrm{d}x, \quad \forall \xi \in \mathbb{R},$$

with the inverse transform given by

$$\mathcal{F}^{-1}\{\hat{f}(\xi)\} = f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i2\pi\xi x}\, \mathrm{d}\xi, \quad \forall x \in \mathbb{R}.$$

These are heavy formulas and will be unpacked in the next section. Note that, in physics and engineering, it is common practice to replace $2\pi\xi$ with ω and $\hat{f}$ is a function of ω, not ξ.

9 Derivation of the Fourier Transform

We begin with the Complex Fourier Series for a function $f : [-L, L] \to \mathbb{C}$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\mathrm{d}x.$$

Define $\xi_n = \dfrac{n}{2L}$. It follows that

$$2\pi\xi_n = \frac{n\pi}{L} \implies e^{i\frac{n\pi}{L}x} = e^{i2\pi\xi_n x}.$$

We then have a new definition of the Fourier Series for f, given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi\xi_n x},$$

where the c_n's are given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x.$$

Now note that the distance between each ξ_n is given by $\Delta\xi_n = \dfrac{1}{2L}$ since $n \in \mathbb{Z}$. This gives a new formula for c_n,

$$c_n = \Delta\xi_n \int_{-L}^{L} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x.$$

Plugging this into our formula for $f(x)$,

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x\right) e^{i2\pi\xi_n x}\Delta\xi_n.$$

Now we apply the limit as $L \to \infty$. Note that this infinite sum is now actually a Riemann Sum, and can thus be written as the following integral,

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x\right) e^{i2\pi\xi_n x}\,\mathrm{d}\xi_n.$$

Now, simply define:

$$\hat{f}(\xi_n) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x,$$

as the Fourier Transform of a function f, and it easily follows that the Inverse Fourier Transform for a function $\hat{f}$ is:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi_n)e^{i2\pi\xi_n x}\,\mathrm{d}\xi_n.$$

Replace the ξ_n's with ξ (since we are no longer operating over a discrete sum with n as an indexing variable), and we have the Fourier Transform and Inverse Fourier Transform as defined in the last section.

10 Fourier Transform Examples

10.1 Example 1

Find the Fourier Transform, $\mathcal{F}\{f(x)\}$, for

$$f(x) = \begin{cases} 1 & x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{else} \end{cases}.$$

Then, $\mathcal{F}\{f(x)\} = \hat{f}(\xi)$ is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} \mathrm{d}x = \int_{-1/2}^{1/2} e^{-i2\pi\xi x} \mathrm{d}x = \frac{1}{-i2\pi\xi} e^{-i2\pi\xi x} \Big|_{-1/2}^{1/2}$$

$$\hat{f}(\xi) = \frac{-1}{2\pi i\xi} \left(e^{-i\pi\xi} - e^{i\pi\xi}\right) = \frac{1}{\pi\xi} \left(\frac{e^{i\pi\xi} - e^{-i\pi\xi}}{2i}\right).$$

By Euler's formula, we know $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, so

$$\hat{f}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

10.2 Example 2

Find the Fourier Transform, $\mathcal{F}\{f(x)\}$, for $f(x) = e^{-\pi x^2}$.

Setting up our integral to find $\hat{f}(\xi)$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi\xi x} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\pi(x^2 + 2i\xi x)} \mathrm{d}x.$$

Completing the square, we see that

$$x^2 + 2i\xi x = (x + i\xi)^2 + \xi^2.$$

Then,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2 - \pi\xi^2} \mathrm{d}x = e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2} \mathrm{d}x.$$

To evaluate this integral, we can use the following well known result:

$$\int_{-\infty}^{\infty} e^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}}.$$

If we substitute $u = x + i\xi$, since $\mathrm{d}u = \mathrm{d}x$, then, since $a = \pi$, we find that

$$\hat{f}(\xi) = e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2} \mathrm{d}x = e^{-\pi\xi^2}.$$

With this same reasoning, we can show a more general result that $\mathcal{F}\left\{e^{-ax^2}\right\} = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2\xi^2/a}$.

10.3 Example 3

Find the Fourier Transform of $f(x)$ where

$$f(x) = \cos(2\pi f_0 x) \cdot \chi_{[-T,T]}(x),$$

where

$$\chi_{[-T,T]}(x) = \begin{cases} 1 & x \in [-T,T] \\ 0 & \text{else} \end{cases},$$

and f_0 is a given frequency. The indicator function $\chi_{[-T,T]}(x)$ ensures that our function is only non zero for $x \in [-T,T]$, so our Fourier Transform is the integral

$$\hat{f}(\xi) = \int_{-T}^{T} \cos(2\pi f_0 x) e^{-i2\pi\xi x} \mathrm{d}x.$$

Using the well known integral identity that $\displaystyle\int \cos(ax)e^{-bx}\mathrm{d}x = \frac{a\sin(ax) - b\cos(ax)}{a^2+b^2}e^{-bx}$ (which can be derived with repeated integration by parts and recursion),

$$\hat{f}(\xi) = \frac{2\pi f_0 \sin(2\pi f_0 x) - i2\pi\xi\cos(2\pi f_0 x)}{(2\pi f_0)^2 + (i2\pi\xi)^2} e^{-i2\pi\xi x}\Bigg|_{-T}^{T} = \frac{f_0\sin(2\pi f_0 x) - i\xi\cos(2\pi f_0 x)}{2\pi(f_0^2 - \xi^2)} e^{-i2\pi\xi x}\Bigg|_{-T}^{T}.$$

Then, evaluating at T and $-T$ and subtracting,

$$\hat{f}(\xi) = \frac{f_0\sin(2\pi f_0 T) - i\xi\cos(2\pi f_0 T)}{2\pi(f_0^2-\xi^2)} e^{-i2\pi\xi T} + \frac{f_0\sin(2\pi f_0 T) + i\xi\cos(2\pi f_0 T)}{2\pi(f_0^2-\xi^2)} e^{i2\pi\xi T}.$$

Now, using Euler's Formula, we can rewrite each exponential term:

$$\begin{aligned} e^{-i2\pi\xi T} &= \cos(2\pi\xi T) - i\sin(2\pi\xi T), \\ e^{2\pi\xi T} &= \cos(2\pi\xi T) + i\sin(2\pi\xi T). \end{aligned}$$

For "simplicity", call $\omega_1 = 2\pi f_0 T$ and $\omega_2 = 2\pi\xi T$. Then, we have

$$\begin{aligned} 2\pi(f_0^2-\xi^2)\hat{f}(\xi) &= (f_0\sin(\omega_1) - i\xi\cos(\omega_1))(\cos(\omega_2) - i\sin(\omega_2)) \\ &\quad +(f_0\sin(\omega_1) + i\xi\cos(\omega_1))(\cos(\omega_2) + i\sin(\omega_2)) \\ &= f_0\sin(\omega_1)\cos(\omega_2) - if_0\sin(\omega_1)\sin(\omega_2) - i\xi\cos(\omega_1)\cos(\omega_2) \\ &\quad -\xi\cos(\omega_1)\sin(\omega_2) + f_0\sin(\omega_1)\cos(\omega_2) + if_0\sin(\omega_1)\sin(\omega_2) \\ &\quad +i\xi\cos(\omega_1)\cos(\omega_2) - \xi\cos(\omega_1)\sin(\omega_2) \\ &= 2f_0\sin(\omega_1)\cos(\omega_2) - 2\xi\cos(\omega_1)\sin(\omega_2) \\ \implies \hat{f}(\xi) &= \frac{f_0\sin(2\pi f_0 T)\cos(2\pi\xi T) - \xi\cos(2\pi f_0 T)\sin(2\pi\xi T)}{\pi(f_0^2-\xi^2)}. \end{aligned}$$

Now, note that, since $\sin\theta\cos\phi = \frac{1}{2}(\sin(\theta+\phi) + \sin(\theta-\phi))$,

$$\begin{aligned} \sin(2\pi f_0 T)\cos(2\pi\xi T) &= \frac{1}{2}\Big(\sin\big(2\pi T(f_0+\xi)\big) + \sin\big(2\pi T(f_0-\xi)\big)\Big), \\ \cos(2\pi f_0 T)\sin(2\pi\xi T) &= \frac{1}{2}\Big(\sin\big(2\pi T(f_0+\xi)\big) - \sin\big(2\pi T(f_0-\xi)\big)\Big). \end{aligned}$$

This then means we have that

$$\hat{f}(\xi) = \frac{f_0 \sin(2\pi T(f_0+\xi)) + f_0 \sin(2\pi T(f_0-\xi)) - \xi \sin(2\pi T(f_0+\xi)) + \xi \sin(2\pi T(f_0-\xi))}{2\pi(f_0^2-\xi^2)}.$$

We can then factor our expression to show that

$$\hat{f}(\xi) = \frac{\sin(2\pi T(f_0+\xi))(f_0-\xi) + \sin(2\pi T(f_0-\xi))(f_0+\xi)}{2\pi(f_0+\xi)(f_0-\xi)} = \frac{\sin(2\pi T(f_0+\xi))}{2\pi(f_0+\xi)} + \frac{\sin(2\pi T(f_0-\xi))}{2\pi(f_0-\xi)}.$$

And we have now taken the Fourier Transform of the nice-looking $f(x)$ to give this horrendous $\hat{f}(\xi)$. As an exercise, compute $\hat{f}(\xi)$ using the fact that

$$\hat{f}(\xi) = \int_{-T}^{T} \cos(2\pi f_0 x) e^{-i2\pi\xi x} \mathrm{d}x = \int_{-T}^{T} \frac{e^{i2\pi f_0 x} + e^{-i2\pi f_0 x}}{2} e^{-i2\pi\xi x} \mathrm{d}x.$$

This will yield the same answer and should work out much easier than the method used above.

The plot of this Fourier Transform is quite interesting. A plot of $\hat{f}(\xi)$ with (arbitrarily chosen) $f_0 = 2.67$ and $T = 4$ is shown below.

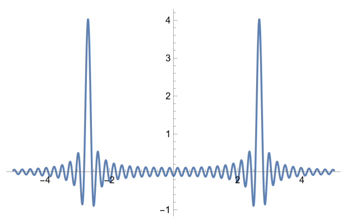

Figure 4: A plot of $\hat{f}(\xi)$ with $f_0 = 2.67$ and $T = 4$.

10.4 Example 4

As an example of the Inverse Fourier Transform, we will take the Inverse Fourier Transform, $\mathcal{F}^{-1}\{\hat{f}(\xi)\} = f(x)$, of $\hat{f}(\xi)$ from our second example,

$$\hat{f}(\xi) = e^{-\pi\xi^2}.$$

Recall that the Inverse Fourier Transform is given by

$$\mathcal{F}^{-1}\{\hat{f}(\xi)\} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x} \, \mathrm{d}\xi.$$

Because $\mathcal{F}^{-1}$ undoes $\mathcal{F}$, we expect the result

$$\mathcal{F}^{-1}\{\hat{f}(\xi)\} = f(x) = e^{-\pi x^2}.$$

We begin by setting up our integral,

$$\mathcal{F}^{-1}\{\hat{f}(\xi)\} = \int_{\mathbb{R}} e^{-\pi\xi^2} e^{i2\pi\xi x} \, \mathrm{d}\xi = \int_{\mathbb{R}} e^{-\pi(\xi^2 - 2ix\xi)} \, \mathrm{d}\xi.$$

Completing the square,

$$\xi^2 - 2ix\xi = (\xi - ix)^2 - (ix)^2 = (\xi - ix)^2 + x^2.$$

Substituting and using the fact that $\displaystyle\int_{\mathbb{R}} e^{-ax^2} = \sqrt{\tfrac{\pi}{a}}$ along with a u-substitution, we see that

$$\mathcal{F}^{-1}\{\hat{f}(\xi)\} = \int_{\mathbb{R}} e^{-\pi(\xi-ix)^2-\pi x^2}\,\mathrm{d}\xi = e^{-\pi x^2}\int_{\mathbb{R}} e^{-\pi(\xi-ix)^2}\,\mathrm{d}\xi = e^{-\pi x^2},$$

as expected.

While this Inverse Transform was fairly straightforward, many Inverse Transforms are not. They often require advanced calculus techniques, which is why this is the example chosen.

11 Meaning of the Fourier Transform

While all of this is well and good, we haven't quite seen why this is an important idea yet. For now, we will restrict ourselves to the investigation of real valued functions, $f : \mathbb{R} \to \mathbb{R}$, under the Fourier Transform. In this case, the Fourier Transform always produces a symmetric function, so we will only consider the transform of a function f over the non-negative reals, $\hat{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Let's return to our third example of Fourier Transforms,

$$f(x) = \cos(2\pi f_0 x)\cdot \chi_{[-T,T]}(x), \quad \hat{f}(\xi) = \frac{\sin(2\pi T(f_0+\xi))}{2\pi(f_0+\xi)} + \frac{\sin(2\pi T(f_0-\xi))}{2\pi(f_0-\xi)}.$$

The reader should use Desmos, Mathematica, MatLab, GeoGebra, or a similar tool and vary T and f_0 to see how $\hat{f}$ changes. Code to do so in Mathematica is provided below.

```
Clear[f0, f, x, xi, T, FT]
f[x_] := Cos[2 Pi f0 x] * Boole[Abs[x] <= T];
FT[xi_] = FourierTransform[f[x], x, xi, FourierParameters -> {0, -2 Pi}];
g[xi_] = Simplify[FT[xi]]

f0 = 3;
T = 3;
Plot[g[xi], {xi, 0, 5}, PlotRange -> All]
Plot[f[x], {x, 0, 5}, PlotRange -> All]
```

The `Manipulate[]` function may also be of use, but typically results in lower performance.

Note that as we change T, what's affected is the domain that $f(x)$ that produces a non-zero output is (since changing T affects where $\chi_{[-T,T]}(x)$ makes the function 0. It also changes the density of the waves and height of the peak in the $\hat{f}(\xi)$ function, with higher T corresponding to a higher density and a higher peak. Some example plots are shown on the next page.

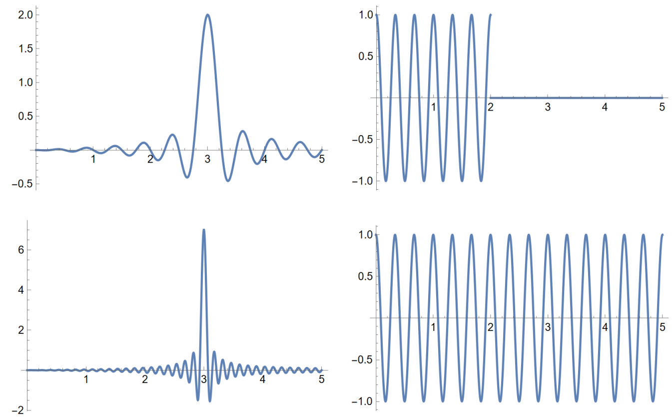

Figure 5: Two plots of $\hat{f}$ and f. The first row has $f_0 = 3$ and $T = 2$, while the second row has $f_0 = 3$ and $T = 7$.

We can see that as T increases, the Fourier Transform seems more "sure of itself"—the waves are much tighter and die down quicker away from the peak, and the peak is at a higher y value. This consequence of the Fourier Transform is very useful and even helps to provide intuition for Heisenberg's Uncertainty Principle in Quantum Physics.

Next, if we vary f_0, we see that this changes (in the graph of f) the length of the period and (in the graph of $\hat{f}$) the location of the peak. Some examples are shown below with T held at 7. We

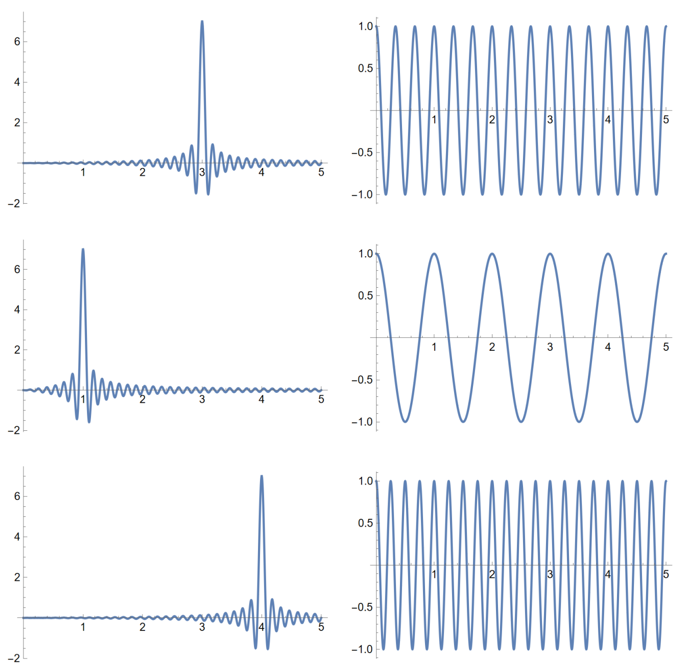

Figure 6: Plots of $\hat{f}$ and f. The first row has $f_0 = 3$, the second has $f_0 = 1$, and the third has $f_0 = 4$.

may observe that the frequency of the original cosine wave has a large effect on the graph of the Fourier Transform. For now, we will focus on varying the frequency of various functions to see how the Fourier Transform is affected. We will hold T at 7. For our next example, consider the function

$$f(x) = \Big(\cos(2\pi f_0 x) + \cos(2\pi f_1 x)\Big) \cdot \chi_{[-T,T]}(x).$$

If you feel so inclined, practice taking a Fourier Transform on this function. The answer will be quite similar to that produced in problem 3. Since we are simply looking at the graphical outputs, we won't worry about doing so here. A few plots are shown below. We see that the plots of f

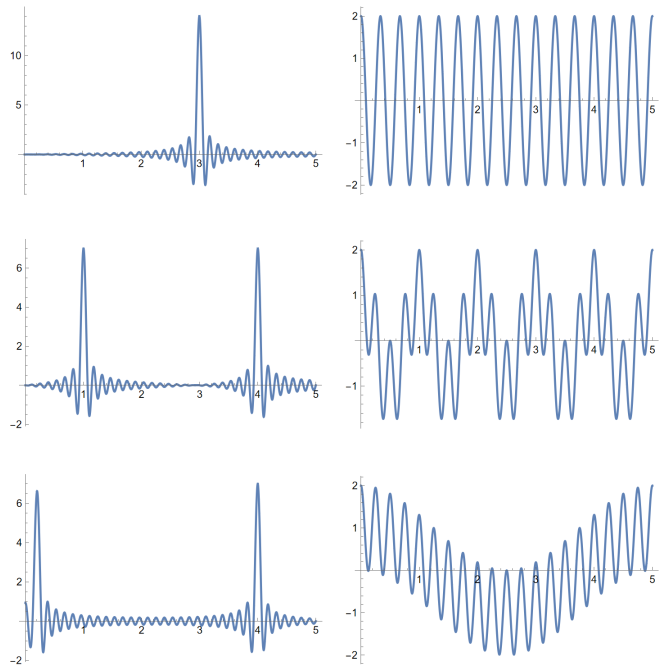

Figure 7: Plots of $\hat{f}$ and f with $T = 7$. The first row has $f_0 = f_1 = 3$, the second row has $f_0 = 1, f_1 = 4$, and the third row has $f_0 = 4, f_1 = 0.2$.

can get quite complex when adding cosine functions of different frequencies. However, the Fourier Transform remains simple and lets us know which frequencies make up the function. But that's not all the Fourier Transform tells us! Let's consider another, slightly more complex, function,

$$f(x) = \Big(3\cos(2\pi f_0 x) - 2\cos(2\pi f_1 x) + \cos(2\pi f_2 x)\Big)\chi_{[-T,T]}(x).$$

Here, we will hold T at 8. This means the peaks will be much more extreme and the waves far more dense, so we will instead plot $\frac{1}{4}\hat{f}(\xi)$ with various frequencies to make things easier to read. Before plotting, let's think about what we should expect to see. As before, there will be larger waves around the frequency values f_0, f_1, and f_2. To further our expectations, note that the Fourier Transform is defined with an integral, a linear function. This means that if we have the Fourier Transform of the sum of some functions, eg. $\mathcal{F}\{f(x)\}$ for $f(x) = g(x) + h(x)$, we can "split up" the Fourier Transform using properties of linearity:

$$\mathcal{F}\{g(x) + h(x)\} = \mathcal{F}\{g(x)\} + \mathcal{F}\{h(x)\}.$$

Another important property of linearity, is that if we take the transform of some function times a scalar value, eg. $\mathcal{F}\{k \cdot f(x)\}$ for some scalar k, we can pull the constant out of the transform using properties of linearity:

$$\mathcal{F}\{k \cdot f(x)\} = k \cdot \mathcal{F}\{f(x)\}.$$

With this, we now actually know what the Fourier Transform for $f(x)$ as defined above will be! We can see that

$$\mathcal{F}\{f(x)\} = 3\mathcal{F}\{\cos(2\pi f_0 x)\chi_{[-T,T]}(x)\} - 2\mathcal{F}\{\cos(2\pi f_1 x)\chi_{[-T,T]}(x)\} + \mathcal{F}\{\cos(2\pi f_2 x)\chi_{[-T,T]}(x)\}.$$

With this, we can conclude that at a frequency of f_0, the height of the corresponding peak will be three times as large as that of the peak corresponding to f_2. We can also conclude that at a frequency of f_1, the "peak" will not actually be a peak, but a valley, since we have a -2 out front. We can also conclude that there will only be one peak if $f_0 = f_1 = f_2$ since we would then just have $2\mathcal{F}\{\cos(2\pi f_0 x)\chi_{[-T,T]}(x)\}$ (by adding the three terms together after replacing f_1 and f_2 with f_0). Let's now take a look at a few plots of this. Again, T is held at 8 and plot $\frac{1}{4}\hat{f}(\xi)$.

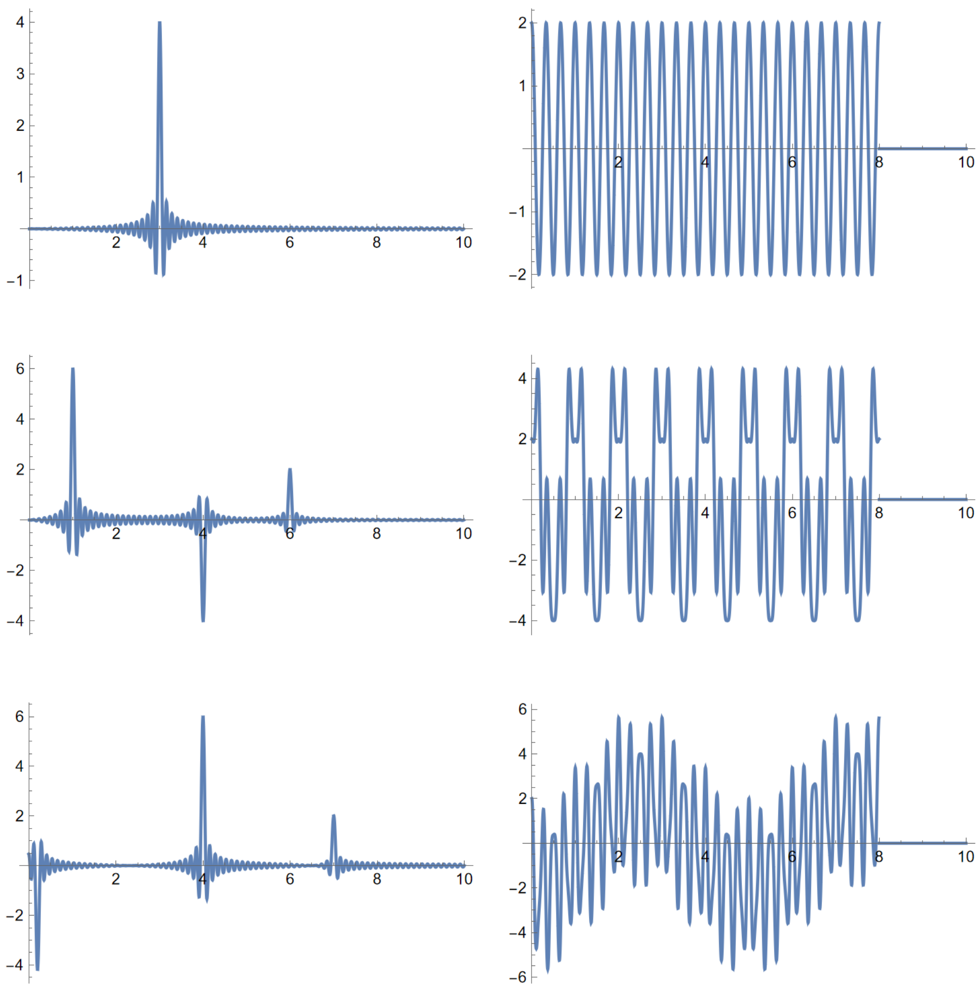

Figure 8: Plots of $\frac{1}{4}\hat{f}(\xi)$ and $f(x)$. The first row has $f_0 = f_1 = f_2 = 3$, the second row has $f_0 = 1, f_1 = 4, f_2 = 6$, and the third row has $f_0 = 4, f_1 = 0.2, f_2 = 7$.

As expected, we see peaks (or valleys) at each frequency.

11.1 Why does any of this matter?

One is right to wonder how this function decomposition has usefulness in the real world, but it is actually one of the most useful concepts in mathematics. From the concept of Fourier Transforms comes the Discrete Fourier Transform (DFT), which leads to the Fast Fourier Transform (FFT). Though they will not be discussed in detail here, they are essentially the Fourier Transform for a function with a non-continuous domain. This is especially useful in the real world, since most data does not have a continuous domain. These transforms are used in signal processing, digital recording, pitch correction, data compression/storing information (JPEG, MP3, and other file types), numerical solutions to ODEs and PDEs, and more. Since the Fourier Transform is more abstract, it's not quite as easily applicable to many of these things; however, it is an extremely useful tool in theoretical work in the fields of ODEs and PDEs, where numerical methods are not often used.

One simple, practical application of the Fourier Transform is to think about a sound recording with an annoying, high pitched background noise. A high-pitched noise will have a higher frequency, so, in a plot of the Fourier Transform (or, more likely, a DFT or FFT), it will be easily identifiable as one of the peaks furthest down the ξ-axis. One can then "remove" this peak and take an Inverse Transform to make a recording with the same sounds except for that annoying background sound.

11.2 Relation between Fourier Transform and Fourier Series

Consider a function $f : \mathbb{R} \to \mathbb{C}$ where f is only non-zero on an interval $[-L, L]$ and let $f_p : \mathbb{R} \to \mathbb{C}$ be the periodic extension of f (by means of Fourier Series). The Fourier Transform of f is given by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}\mathrm{d}x = \int_{-L}^{L} f(x)e^{-i2\pi\xi x}\mathrm{d}x,$$

since $f(x) = 0$ outside of the interval $[-L, L]$. The complex Fourier coefficients of f_p are given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\mathrm{d}x.$$

Recall that $\xi_n = \frac{n}{2L} \implies 2\pi\xi_n = \frac{n\pi}{L}$. Then, we see that c_n is given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i2\pi\xi_n x}\mathrm{d}x = \frac{1}{2L}\hat{f}(\xi_n).$$

Framed differently,

$$c_n = \frac{1}{2L}\hat{f}\left(\frac{n}{2L}\right).$$

As framed here, this is quite abstract. Seen in an example, this will make more sense. Consider the function $f(x) = \chi_T(x) = \chi_{[-T,T]}(x)$ for some T. Remember that

$$\chi_{[-T,T]}(x) = \begin{cases} 1 & x \in [-T, T] \\ 0 & \text{else} \end{cases}.$$

Taking the Fourier Transform,

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}\mathrm{d}x = \int_{-T}^{T} e^{-i2\pi\xi x}\mathrm{d}x = \frac{1}{-i2\pi\xi}(e^{-i2\pi\xi T} - e^{i2\pi\xi T}) = \frac{\sin(2\pi\xi T)}{\pi\xi}.$$

To find the Fourier Series, we must define a period, L. In order for this function to actually have a period, we must take $L > T$. If $L \leq T$, the function will never be 0, and will then just be the line $y = 1$, which is not of interest. For $L > T$, we can find the complex Fourier coefficients c_n by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\mathrm{d}x = \frac{1}{2L}\int_{-T}^{T} e^{-i\frac{n\pi}{L}x}\mathrm{d}x = \frac{-1}{2i\pi n}(e^{-i\frac{n\pi}{L}T} - e^{i\frac{n\pi}{L}T}) = \frac{\sin(\frac{n\pi}{L}T)}{n\pi}.$$

Then, if we evaluate $\hat{f}(\xi_n) = \hat{f}(\frac{n}{2L})$, we have

$$\hat{f}(\xi_n) = \frac{\sin(\frac{n\pi}{L}T)}{\pi\frac{n}{2L}} = \frac{2L\sin(\frac{n\pi}{L}T)}{\pi n} = 2Lc_n \implies c_n = \frac{1}{2L}\hat{f}(\xi_n).$$

11.3 Why is the Fourier Transform so useful in the study of Differential Equations?

Let's begin by taking the Fourier Transform of a general $f(x)$ and its derivative $f'(x)$. For $f(x)$, its transform is given by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}\mathrm{d}x.$$

For $f'(x)$, its transform is given similarly by

$$\mathcal{F}\{f'(x)\} = \hat{f}'(\xi) = \int_{\mathbb{R}} f'(x)e^{-i2\pi\xi x}\mathrm{d}x.$$

Integrating by parts,

$$\hat{f}'(\xi) = f(x)e^{-i2\pi\xi x}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}(-i2\pi\xi)\mathrm{d}x = f(x)e^{-i2\pi\xi x}\Big|_{-\infty}^{\infty} + i2\pi\xi\int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}\mathrm{d}x.$$

For $f(x)$ meeting certain requirements (discussed shortly), the boundary term $f(x)e^{-i2\pi\xi x}\Big|_{-\infty}^{\infty}$ vanishes ($= 0$), and we are left with

$$\mathcal{F}\{f'(x)\} = i2\pi\xi\hat{f}(\xi).$$

Amazingly, like the Laplace Transform, the Fourier Transform relates a derivative to its parent function in an algebraic way, allowing for algebraic solutions to a differential equation. Once an algebraic solution is found, one can take the inverse transform and find $f(x)$ solving a given differential equation.

Let us now discuss the limitations of $f(x)$ where this applies. Because $e^{-i2\pi\xi x}$ is a complex number with modulus 1 (the modulus of a complex number z is given by $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$), then, as long as $\lim_{x\to\pm\infty} f(x) = 0$, the limit of the product will be 0 as $x \to \pm\infty$ and thus the boundary term vanishes. Another way to think about why this is true is to remember Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. This means that as $x \to \pm\infty$, the $e^{-i2\pi\xi x}$ term simply oscillates, so the limit $\lim_{x\to\pm\infty} f(x)e^{-i2\pi\xi x}$ is dominated by the $f(x)$ term. Thus, so long as $f \to 0$ at the infinities, the boundary term vanishes.

11.3.1 Example (ODE)

Find a general solution to the second-order differential equation

$$y''(x) + y(x) = f(x) \quad \lim_{x\to\pm\infty} y(x) = 0.$$

The Fourier Transform of $y(x)$ and $f(x)$ are given by $\hat{y}(\xi)$ and $\hat{f}(\xi)$, respectively. For y'',

$$\mathcal{F}\{y''\} = -4\pi^2\xi^2\hat{y}.$$

Now, we have the following corresponding algebraic equation in ξ:

$$-4\pi^2\xi^2\hat{y}(\xi) + \hat{y}(\xi) = \hat{f}(\xi).$$

Here, we can factor out a $\hat{y}(\xi)$ then divide by $1 - 4\pi^2\xi^2$, giving us

$$\hat{y}(\xi) = \frac{\hat{f}(\xi)}{1 - 4\pi^2\xi^2}.$$

We can now find $y(x)$ by taking the inverse transform of this equation. To take the inverse transform of the right hand side of this equation, one must use the *Convolution Theorem*, which will not be covered here. Most inverse transforms use somewhat advanced methods, and, as such, we cannot hope to cover all of the methods of taking an Inverse Fourier Transform in this document.

11.3.2 Example (PDE)

Consider the wave equation in one dimension over all of $\mathbb{R}$,

$$\begin{cases} \partial_t^2 u(x,t) = \partial_x^2 u(x,t), \\ u(x,0) = f(x), \\ \partial_t(x,0) = g(x). \end{cases}$$

We can take the Fourier Transform with respect to the x variable, giving us

$$\begin{cases} \hat{u}_{tt}(\xi,t) = -4\pi^2\xi^2\hat{u}(\xi,t), \\ \hat{u}(\xi,0) = \hat{f}(\xi), \\ \hat{u}_t(\xi,0) = \hat{g}(\xi), \end{cases}$$

where subscripts denote partial derivatives. For each fixed ξ, this leaves behind a second order ODE in t, which can be solved by various methods. Of these, the simplest is to notice that we have a function such that when we take its derivative twice, we get a negative constant times that function. We know two functions that meet this requirement, those being $\sin(t)$ and $\cos(t)$. Thus, we know our solution will be of the form

$$\hat{u}(\xi,t) = c_1\cos(2\pi\xi t) + c_2\sin(2\pi\xi t),$$

for some scalars c_1, c_2. We can solve for these scalars by applying the initial conditions,

$$\hat{u}(\xi,0) = \hat{f}(\xi) = c_1\cos(0) + c_2\sin(0) \implies c_1 = \hat{f}(\xi),$$
$$\hat{u}_t(\xi,0) = \hat{g}(\xi) = -2\pi\xi c_1\sin(0) + 2\pi\xi c_2\cos(0) \implies c_2 = \frac{\hat{g}(\xi)}{2\pi\xi}.$$

Thus, our solution is

$$\hat{u}(\xi, t) = \hat{f}(\xi)\cos(2\pi\xi t) + \frac{\hat{g}(\xi)}{2\pi\xi}\sin(2\pi\xi t).$$

To find $u(x, t)$, we must take the inverse transform of $\hat{u}(\xi, t)$ with respect to ξ. Again, this will not be covered here. These examples are just to give an idea of how the Fourier Transform can be used to help solve differential equations.

Further Readings

[1] Wolfram MathWorld. *Inner Product.* https://mathworld.wolfram.com/InnerProduct.html

[2] Auburn University. *Inner Products.* https://web.auburn.edu/holmerr/2660/Textbook/innerproduct-print.pdf

[3] Tarquino, J. *Fourier Series / Transform Introduction.* https://math.uchicago.edu/~may/REU2023/REUPapers/Tarquino.pdf

[4] Xue, S. *Convergence of Fourier Series.* https://math.uchicago.edu/~may/REU2017/REUPapers/Xue.pdf

[5] University of Arizona. *Fourier Transform.* https://math.arizona.edu/~kglasner/math456/fouriertransform.pdf

[6] Harvard University. *Fourier Transform.* https://scholar.harvard.edu/files/schwartz/files/lecture8-fouriertransforms.pdf

[7] University of Chicago. *PDEs and the Fourier Transform.* https://math.uchicago.edu/~luis/preprints/pdenotes.pdf

[8] Wolfram MathWorld. *Convolution.* https://mathworld.wolfram.com/Convolution.html

[9] Wolfram MathWorld. *Convolution Theorem.* https://mathworld.wolfram.com/ConvolutionTheorem.html

[10] Oxford University. *Discrete Fourier Transform.* https://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf

[11] Wolfram MathWorld. *Discrete Fourier Transform.* https://mathworld.wolfram.com/DiscreteFourierTransform.html

[12] Wolfram MathWorld. *Fast Fourier Transform.* https://mathworld.wolfram.com/FastFourierTransform.html

[13] Carnegie Mellon University. *Fast Fourier Transform.* https://www.cs.cmu.edu/afs/andrew/scs/cs/15-463/2001/pub/www/notes/fourier/fourier.pdf