Rudin Theorems 7.9, 7.17, and 7.26

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This paper focuses on Chapter 7 of Rudin's *Principles of Mathematical Analysis*. Here, we present a proof for Theorem 7.9, which was omitted, present the proof suggested in the remark following the proof of Theorem 7.17, give an improvement to the proof of Theorem 7.26, the Weierstrass Theorem, and give sketches of a few of the Q_n from the proof of Theorem 7.26.

Theorem 7.9 (Rudin)**.** *Suppose*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof. First, suppose $f_n \to f$ uniformly on E and fix an $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $n \geq N$ implies that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$. Taking the supremum,

$$M_n = \sup_{x\in E} |f_n(x) - f(x)| \leq \varepsilon.$$

Thus $M_n \to 0$ as $n \to \infty$.

Conversely, suppose $M_n \to 0$ as $n \to \infty$. Then, for any $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ so that for $n \geq N$, $|f_n(x) - f(x)| \leq M_n \leq \varepsilon$. Thus $f_n \to f$ uniformly on E. □

Theorem 7.17 (Rudin)**.** *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \leq x \leq b). \tag{7.1}$$

After proving this theorem, Rudin remarks that,

> If the continuity of the functions f_n' is assumed in addition to the above hypotheses, then a much shorter proof of (7.1) can be based on Theorem 7.16 and the fundamental theorem of calculus.

For completeness' sake, we restate Theorem 7.16, then prove Theorem 7.17 with the additional hypothesis.

Theorem 7.16 (Rudin)**.** *Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and*

$$\int_a^b f \ \mathrm{d}\alpha = \lim_{n\to\infty} \int_a^b f_n \ \mathrm{d}\alpha.$$

Theorem 7.17 (Additional Hypothesis)**.** *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$ and f_n' is continuous on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b).$$

Proof. In the setup of Theorem 7.16, let $\alpha(x) = x$. Then $f_n' \in \mathscr{R}$ on $[a,b]$. Let g denote the function such that $f_n' \to g$ uniformly on $[a,b]$. Then also $g(x) = \lim_{n\to\infty} f_n'(x)$. By Theorem 7.16, $g \in \mathscr{R}$ on $[a,b]$ and, when $x \in [a,b]$,

$$\lim_{n\to\infty} \int_a^x f_n'(t) \ \mathrm{d}t = \int_a^x g(t) \ \mathrm{d}t.$$

Now, by the fundamental theorem of calculus, since each f_n' is continuous,

$$f_n(x) = f_n(a) + \int_a^x f_n'(t) \ \mathrm{d}t.$$

Take $x = x_0$. We then have that

$$f_n(x_0) = f_n(a) + \int_a^{x_0} f_n'(t) \ \mathrm{d}t.$$

By Theorem 7.16, the sequence of integrals

$$\left\{\int_a^{x_0} f_n'(t) \ \mathrm{d}t\right\} \to \int_a^{x_0} g(t) \ \mathrm{d}t.$$

Then $f_n(a)$ is equal to the difference of two convergent sequences, and thus convergent. Call $c = \lim_{n\to\infty} f_n(a)$. Define

$$f(x) := c + \int_a^x g(t) \ \mathrm{d}t$$

so that, by the fundamental theorem of calculus, $f'(x) = g(x)$. Then

$$\begin{aligned} f_n(x) - f(x) &= f_n(a) + \int_a^x f_n'(t) \ \mathrm{d}t - c - \int_a^x g(t) \ \mathrm{d}t \\ &= f_n(a) - c + \int_a^x f_n'(t) - g(t) \ \mathrm{d}t. \end{aligned}$$

Applying the triangle inequality,

$$|f_n(x) - f(x)| \le |f_n(a) - c| + \left| \int_a^x f_n'(t) - g(t) \; \mathrm{d}t \right| . \tag{7.2}$$

Set $h_n(t) = f_n'(t) - g(t)$. Note that

$$\left| \int_a^x h_n(t) \; \mathrm{d}t \right| \le \int_a^x |h_n(t)| \, \mathrm{d}t \le \int_a^b |h_n(t)| \, \mathrm{d}t \le (b-a) \sup_{t \in [a,b]} |h_n(t)| .$$

Taking supremums over $x \in [a, b]$, we see that

$$\sup_{x \in [a,b]} \left| \int_a^x f_n'(t) - g(t) \; \mathrm{d}t \right| \le (b-a) \sup_{t \in [a,b]} |f_n'(t) - g(t)| .$$

Since $f_n' \to g$ uniformly, then as $n \to \infty$, also

$$\sup_{x \in [a,b]} \left| \int_a^x f_n'(t) - g(t) \; \mathrm{d}t \right| \to 0.$$

Fix an $\varepsilon > 0$. As $n \to \infty$ in (7.2) after taking supremums over $x \in [a, b]$, $|f_n(a) - c| \to 0$ and so does the second term (as above), and, thus, there is an $N \in \mathbb{N}$ so that for $n \ge N$, we get that $|f_n(x) - f(x)| \le \varepsilon$ for all $x \in [a, b]$, so $f_n \to f$ uniformly on $[a, b]$. Finally, note that $f'(x) = g(x) = \lim_{n \to \infty} f_n'(x)$. □

We now give an improved proof and extra justification for Theorem 7.26. First, a Lemma.

Lemma 1.1 (Bernoulli's Inequality)**.** *For real $b \le 1$ and a positive integer n,*

$$(1-b)^n \ge 1 - nb.$$

Proof. Note that $(1-b)^1 \ge 1 - (1)b$. Suppose that for some $k \in \mathbb{N}$, $(1-b)^k \ge 1 - kb$. Then, since $1 - b \ge 0$,

$$(1-b)^{k+1} \ge (1-kb)(1-b) = 1 - kb - b + kb^2 \ge 1 - (k+1)b.$$

By induction, the Lemma follows. □

Theorem 7.26 (Rudin)**.** *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken as real.

Proof. We may assume, without loss of generality, that $[a, b] = [0, 1]$. If the theorem is proven in this case, note that any $y \in [a, b]$ can be written as $y = a + (b - a)x$ for an $x \in [0, 1]$. Then we can define $g : [0,1] \to \mathbb{C}$ so that $g(x) = f(a + (b-a)x)$. After applying the theorem to g and getting a sequence $P_n(x)$ of polynomials, we see that

$$f(y) = g\left(\frac{y-a}{b-a}\right)$$

so that $P_n\left(\frac{y-a}{b-a}\right)$ is also a sequence of polynomials that converges to $f(y)$. We may also assume that $f(0) = f(1) = 0$. For, if the theorem is proved in this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here, $g(0) = g(1) = 0$, so, if g can be obtained as the limit of a uniformly convergent sequence of polynomials, then the same is true of f since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0,1]$. Since $f(a) = f(b)$ for $a, b \notin (0,1)$, it is clear that f is uniformly continuous on $\mathbb{R} \setminus (0,1)$. Since $[0,1]$ is compact, f is uniformly continuous on $[0,1]$ as well. For each ε, let δ be the corresponding δ in the definition of uniform continuity for f on $[0,1]$. Then, given an ε and two points $x, y \in \mathbb{R}$ so that $|x-y| < \delta$, we see that:

- If $x, y \in [0,1]$, $|f(x) - f(y)| < \varepsilon$,
- If $x, y \notin (0,1)$, then $f(x) = f(y) = 0$, so $|f(x) - f(y)| < \varepsilon$,
- If $x \in [0,1]$ but $y \notin (0,1)$, then either $y \le 0$ or $y \ge 1$. If $y \ge 1$, then, since also $|x-1| < \delta$, we have $|f(x) - f(1)| < \varepsilon \iff |f(x) - 0| < \varepsilon \iff |f(x) - f(y)| < \varepsilon$. The case for $y \le 0$ is handled in the exact same manner.

We put

$$Q_n(x) = c_n(1-x^2)^n \qquad (n = 1, 2, 3, \ldots),$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)\ \mathrm{d}x = 1 \qquad (n = 1, 2, 3, \ldots). \tag{7.3}$$

We need some information about the order of magnitude of c_n. Since

$$\begin{aligned}\int_{-1}^{1} (1-x^2)^n\ \mathrm{d}x = 2\int_{0}^{1} (1-x^2)^n\ \mathrm{d}x &\ge 2\int_{0}^{1/\sqrt{n}} (1-x^2)^n\ \mathrm{d}x \\ &\ge 2\int_{0}^{1/\sqrt{n}} (1-nx^2)\ \mathrm{d}x = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},\end{aligned}$$

where the second inequality follows from Lemma 1.1 with $b = x^2$. It follows from (7.3) that

$$c_n < \sqrt{n}. \tag{7.4}$$

Then when $\delta \in [0, |x|]$, (7.4) gives a bound for Q_n,

$$Q_n(x) \le \sqrt{n}(1-\delta^2)^n \tag{7.5}$$

Note that $f \in \mathscr{R}$ since it is continuous and bounded (Theorem 6.10). Set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\ \mathrm{d}t \qquad (0 \le x \le 1).$$

Our assumptions about f show, by a simple change of variables,

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)\ \mathrm{d}t = \int_{0}^{1} f(u)Q_n(u-x)\ \mathrm{d}u.$$

The last integral is a polynomial in x because, by the binomial formula,

$$\int_0^1 f(u)Q_n(u-x)\ \mathrm{d}u = \int_0^1 f(u)c_n \sum_{k=0}^{n} \binom{n}{k}(-(u-x)^2)^k\ \mathrm{d}u = \int_0^1 f(u)c_n \sum_{k=0}^{n} \binom{n}{k}(-1)^k(u-x)^{2k}.$$

With another application of the binomial formula,

$$\int_0^1 f(u)c_n \sum_{k=0}^{n} \binom{n}{k}(-1)^k(u-x)^{2k} = \int_0^1 f(u)c_n \sum_{k=0}^{n} \binom{n}{k}(-1)^k \sum_{j=0}^{2k} \binom{2k}{j} u^{2k-j}(-1)^j x^j.$$

Since the sums are finite,

$$\int_0^1 f(u)Q_n(u-x)\ \mathrm{d}u = c_n \sum_{k=0}^{n}\sum_{j=0}^{2k} \binom{n}{k}\binom{2k}{j}(-1)^{k+j}x^j \int_0^1 f(u)u^{2k-j}\ \mathrm{d}u.$$

Each $\int_0^1 f(u)u^{2k-j}\ \mathrm{d}u \in \mathbb{C}$, so the remaining expression is a function only of x: a polynomial of degree at most $2n$. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

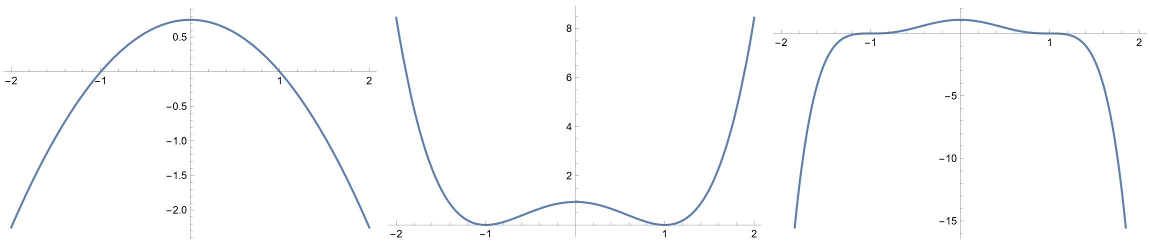
$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (7.3), (7.5), and the fact that $Q_n(x) \ge 0$, we see that, using Theorem 6.13 for the first inequality, for $0 \le x \le 1$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} (f(x+t) - f(x))Q_n(t)\ \mathrm{d}t \right| \\
&\le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\ \mathrm{d}t \\
&\le 2M \int_{-1}^{-\delta} Q_n(t)\ \mathrm{d}t + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\ \mathrm{d}t + 2M \int_{\delta}^{1} Q_n(t)\ \mathrm{d}t \\
&\le 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for large enough n, which proves the theorem. □

Below are plots of the $Q_n(x)$s for $n = 1, 2, 3$ on $[-2, 2]$.



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

References

[1] Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw–Hill, New York, 1976.